Second semestral exam 2018
B.Math. Hons. 2nd year
Algebra IV
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Q 1. (4+6)
Let L_1, L_2 be finite extensions of K contained in an algebraic extension L.
(i) Give an example to show that the composite extension L_1L_2 may satisfy $[L_1L_2 : K] < [L_1 : K][L_2 : K]$.
(ii) If the composite extension L_1L_2 satisfies $[L_1L_2 : K] = [L_1 : K][L_2 : K]$, prove that $L_1 \cap L_2 = K$.

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Q 2.
Let K be a field of positive characteristic p. If there exists $a \in K$ which is not a p-th power in K, prove that there exist irreducible polynomials of degree p^n over K for arbitrary n.

OR

Let $f \in K[X]$ be irreducible and suppose L/K is a splitting field of f over K. Prove that all roots of f in L must have the same multiplicity.

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Q 3.
For a finite field F, show that any irreducible polynomial in $F[X]$ must divide $X^{p^{deg(f)}} - X$ in $F[X]$.

OR

Let L be a field and G be a finite group of automorphisms of L. Prove that L is a Galois extension of L^G.

Q 4.
Determine the largest natural number N such that the degree of the field extension $\mathbb{Q}(\{\sqrt{n} : 1 \leq n \leq N\})$ is less than 1000.

OR

Let $L = K(\alpha)$ where α is algebraic over K. Show that there are only finitely many intermediate fields between K and L.

Q 5.
Let $f \in K[X_1, \cdots, X_n]$ be irreducible. Consider the field of fractions L of the integral domain $K[X_1, \cdots, X_n]/(f)$. If the X_1-degree of f is positive, find a subset of $\{X_1, \cdots, X_n\}$ whose image in L forms a transcendence base of L over K.

OR

Show that $\mathbb{C}$ has uncountably many different proper subfields over which $\mathbb{C}$ is of degree 2.